Problem Set 1[1]
Math Camp, Summer 2023, UCSB
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Due: 11:59 PM on Tuesday, Aug 29, 2023

This problem set will help you review the key concepts from the course so far. You are asked to submit this assignment using LaTeX or R markdown (only for the Linear Algebra problem sets). The purpose is to give you an opportunity to be exposed to LaTeX and R Markdown if you haven't. I will give you a brief lecture about R markdown this Friday, so please spend time for solving problems by hand before then. Please have only one member in your group send an email to the instructor by the due date with other members cc'ed in the email.

1. Consider the following matrices:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) Check $(\mathcal{A}\mathcal{B})^T = \mathcal{B}^T \mathcal{A}^T$.
   (b) Check $(\mathcal{A}\mathcal{B})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$.
   (c) Check $(\mathcal{A}^T)^{-1} = (\mathcal{A}^{-1})^T$.

2. Let A and B be $n \times n$ matrices, where n is a positive integer ($n \geq 1$). Prove whether the following is true or false:

$$\det(A+B) = \det(A) + \det(B)$$

3. $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{W} \subset \mathbb{R}^n$ forms a basis of $\mathbb{W}$. Then for a vector $\mathbf{w} \in \mathbb{W}$, prove that there exists a "*unique*" $[c_1 \;\cdots\; c_k]' \in \mathbb{R}^k$ such that

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{v}_i$$

   Hint: You may start by supposing it isn't.

4. Let X be a $n \times k$ matrix with $\text{rank}(\mathbf{X}) = k (n \geq k)$. An annihilator matrix $\mathcal{M}_{\mathbf{X}}$ is

$$\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

   Show that $\mathcal{M}_{\mathbf{X}}$ is symmetric and idempotent. Clarify which property you are using in each step.

5. Find the eigenvalues and eigenvectors of the following matrix. Normalize the norm to one.

$$\mathcal{C} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$$

[1]This problem set is created by Woongchan Jeon and modified by Seonmin (Will) Heo.

6. Express $-\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2}$ and $\sum_{i=1}^{n} \lambda_i u_i^2$ into quadratic forms using

$$\mathbf{u} := [u_1 \ \cdots \ u_n]', \quad \Sigma := \sigma^2 \mathcal{I}_n, \quad \text{and} \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

7. Consider the following regression equation where $n > k$:

$$\begin{aligned} y_i &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ik-1}\beta_{k-1} + \beta_k + u_i && \forall \ i = 1, \cdots, n \\ &= \mathbf{x}_i{}^T \boldsymbol{\beta} + u_i && \forall \ i = 1, \cdots, n \end{aligned}$$

We can rewrite this as matrix equation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

(a) Check the followings: $\mathbf{X}^T\mathbf{X} = \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T$ and $\mathbf{X}^T\mathbf{y} = \sum_{i=1}^{n} \mathbf{x}_i y_i$.

(b) Assume that $\text{rank}(\mathbf{X}) = k$. We derived OLS estimator $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ in class. Show that

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i\right).$$

HINT: For any vector $\widehat{\boldsymbol{\beta}}$ and $n \in \mathbb{N}$, $1\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}$ and $1 = n^{-1}n$.

(c) Show that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T\right)^{-1} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{u}_i\right)$$

HINT: You may start from what you've shown in (b) and replacing y_i with $x_i^T\beta + u_i$.

(d) Let $\hat{\mathbf{u}} := \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$. Show that $\sum_{i=1}^{n} \hat{u}_i^2 = \mathbf{u}^T \mathcal{M}_{\mathbf{X}} \mathbf{u}$ where $\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$. HINT: You may start from

$$\hat{\mathbf{u}} = \mathcal{M}_{\mathbf{X}}\mathbf{y} = \mathcal{M}_{\mathbf{X}}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u})$$

8. This is a problem using R.

(a) Set up a function in R to determine the singularity of the matrix in each matrix equation below. The function should automatically derive a solution if it is non-singular, or should print "It does not have a unique solution" if a matrix is singular. (use `det()` function to derive the determinants of matrices)

i.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

ii.

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

(b) Create another function including a for loop to derive the determinant of the matrix using cofactor(), which means this time you should use Laplace Expansion instead of det(). Compare it with the value derived from det() function from each matrix equation in (a).