Problem Set 1[1]
Math Camp, Summer 2023, UCSB
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1. Consider the following matrices:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) Check $(\mathcal{A}\mathcal{B})^T = \mathcal{B}^T\mathcal{A}^T$.

$$(\mathcal{A}\mathcal{B})^T = \begin{bmatrix} 14 & 69 \\ 4 & 30 \end{bmatrix}; \quad \mathcal{B}^T\mathcal{A}^T = \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 69 \\ 4 & 30 \end{bmatrix}$$

   (b) Check $(\mathcal{A}\mathcal{B})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$.

$$(\mathcal{A}\mathcal{B})^{-1} = \frac{1}{144}\begin{bmatrix} 30 & -4 \\ -69 & 14 \end{bmatrix};$$

$$\mathcal{B}^{-1}\mathcal{A}^{-1} = \frac{1}{9}\cdot\frac{1}{16}\begin{bmatrix} 3 & -2 \\ -6 & 7 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ -3 & 2 \end{bmatrix} = \frac{1}{144}\begin{bmatrix} 30 & -4 \\ -69 & 14 \end{bmatrix}$$

   (c) Check $(\mathcal{A}^T)^{-1} = (\mathcal{A}^{-1})^T$.

$$(\mathcal{A}^T)^{-1} = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} 8 & -3 \\ 0 & 2 \end{bmatrix} \quad (\mathcal{A}^{-1})^T = \frac{1}{16}\begin{bmatrix} 8 & -3 \\ 0 & 2 \end{bmatrix}$$

2. Let A and B be $n \times n$ matrices, where n is a positive integer ($n \geq 1$). Prove whether the following is true or false:

$$\det(A+B) = \det(A) + \det(B)$$

False.
Proof. Consider the following two matrices

$$\mathcal{A} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix},$$

where k is a non-zero integer. Then $\det(A) = 0$ and $\det(B) = 0$, so $\det(A) + \det(B) = 0$. However, $\det(A+B) = \det(k \cdot \mathcal{I}_2) = k^2$, which is non-zero. ■

[1] This problem set is created by Woongchan Jeon and modified by Seonmin (Will) Heo.

3. $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{W} \subset \mathbb{R}^n$ forms a basis of $\mathbb{W}$. Then given $\mathbf{w} \in \mathbb{W}$, prove that there exists a "*unique*" $[c_1 \ \cdots \ c_k]' \in \mathbb{R}^k$ such that

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{v}_i$$

HINT: You may start by supposing it isn't.

Proof. Suppose not. Then there exists $\mathbf{c}, \mathbf{d} \in \mathbb{R}^k$ where $\mathbf{c} \neq \mathbf{d}$ such that

$$\mathbf{w} = \sum_{i=1}^{k} c_i \mathbf{v}_i = \sum_{i=1}^{k} d_i \mathbf{v}_i$$

This implies

$$\sum_{i=1}^{k} (c_i - d_i) \mathbf{v}_i = \mathbf{0}_n$$

This contradicts to linear independence since there exists a nontrivial solution for the vector equation above. ■

4. Let X be a $n \times k$ matrix with $\text{rank}(\mathbf{X}) = k (n \geq k)$. An annihilator matrix $\mathcal{M}_{\mathbf{X}}$ is

$$\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

Show that $\mathcal{M}_{\mathbf{X}}$ is symmetric and idempotent. Clarify which property you are using in each step.

$\mathcal{M}_{\mathbf{X}}$ is symmetric:

$$\begin{aligned}
\mathcal{M}_{\mathbf{X}}^T &= (\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)^T \\
&= \mathcal{I}_n - (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)^T && \mathcal{I}_n^T = \mathcal{I}_n \\
&= \mathcal{I}_n - \mathbf{X}((\mathbf{X}^T\mathbf{X})^{-1})^T\mathbf{X}^T && (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \\
&= \mathcal{I}_n - \mathbf{X}((\mathbf{X}^T\mathbf{X})^T)^{-1}\mathbf{X}^T && (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T \\
&= \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T && (\mathbf{A}^T)^T = \mathbf{A} \\
&= \mathcal{M}_{\mathbf{X}}.
\end{aligned}$$

$\mathcal{M}_{\mathbf{X}}$ is idempotent:

$$\begin{aligned}
\mathcal{M}_{\mathbf{X}}\mathcal{M}_{\mathbf{X}} &= (\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) \\
&= \mathcal{I}_n - 2(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) + \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
&= \mathcal{I}_n - 2(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) + \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T && (\text{cancel out } \mathbf{X}^T\mathbf{X}) \\
&= \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
&= \mathcal{M}_{\mathbf{X}}.
\end{aligned}$$

5. Find eigenvalues and eigenvectors of the following matrix. Normalize the norm to one.

$$\mathcal{C} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$$

$$\begin{aligned} |(\mathcal{C} - \lambda I_2)| &= \begin{vmatrix} \frac{4}{5} - \lambda & \frac{1}{20} \\ \frac{1}{5} & \frac{19}{20} - \lambda \end{vmatrix} \\ &= \left(\frac{4}{5} - \lambda\right)\left(\frac{19}{20} - \lambda\right) - \frac{1}{100} \\ &= \lambda^2 - \frac{7}{4}\lambda + \frac{3}{4} \\ &= (\lambda - 1)\left(\lambda - \frac{3}{4}\right). \end{aligned}$$

Obtaining eigenvectors corresponding to each eigenvalue yields

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix} \quad \text{and} \quad \lambda_2 = \frac{3}{4}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

6. Express $-\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2}$ and $\sum_{i=1}^{n} \lambda_i u_i^2$ into quadratic forms using

$$\mathbf{u} := [u_1 \ \cdots \ u_n]', \quad \Sigma := \sigma^2 \mathcal{I}_n, \quad \text{and} \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$-\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2} = -\frac{1}{2}\mathbf{u}^T \Sigma^{-1} \mathbf{u} \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i v_i^2 = \mathbf{u}^T \Lambda \mathbf{u}$$

7. Consider the following regression equation where $n > k$:

$$\begin{aligned} y_i &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ik-1}\beta_{k-1} + \beta_k + u_i & \forall\ i = 1, \cdots, n \\ &= \mathbf{x}_i{}^T \boldsymbol{\beta} + u_i & \forall\ i = 1, \cdots, n \end{aligned}$$

We can rewrite this as matrix equation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

(a) Check the followings: $\mathbf{X}^T\mathbf{X} = \sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T$ and $\mathbf{X}^T\mathbf{y} = \sum_{i=1}^n \mathbf{x}_i y_i$.

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1k-1} & \cdots & x_{nk-1} \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_{i1}^2 & \sum x_{i1}x_{i2} & \cdots & \sum x_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ \sum x_{i1} & \sum x_{i2} & \cdots & n \end{bmatrix}$$

$$\mathbf{x_i}\mathbf{x_i}^T = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ik-1} \\ 1 \end{bmatrix} \begin{bmatrix} x_{i1} & \cdots & x_{ik-1} & 1 \end{bmatrix} = \begin{bmatrix} x_{i1}^2 & x_{i1}x_{i2} & \cdots & x_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i1} & x_{i2} & \cdots & 1 \end{bmatrix}$$

$$\mathbf{X}^T\mathbf{y} = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1k-1} & \cdots & x_{nk-1} \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum x_{i1}y_i \\ \sum x_{i2}y_i \\ \vdots \\ \sum y_i \end{bmatrix} = \sum_{i=1}^n \mathbf{x}_i y_i.$$

(b) Assume that $\text{rank}(\mathbf{X}) = k$. We derived OLS estimator $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ in class. Show that

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right)$$

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\mathbf{X}^T\mathbf{X}\right)^{-1}\left(\frac{1}{n}\mathbf{X}^T\mathbf{y}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right)$$

(c) Show that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right).$$

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(\mathbf{x}_i^T\boldsymbol{\beta} + \mathbf{u}_i)\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)\boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right)
\end{aligned}$$

(d) Let $\hat{\mathbf{u}} := \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$. Show that $\sum_{i=1}^{n}\hat{u}_i^2 = \mathbf{u}^T\mathcal{M}_{\mathbf{X}}\mathbf{u}$ where $\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$.

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}} \\
&= \mathbf{y} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} \\
&= \left(\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\mathbf{y} \\
&= \mathcal{M}_{\mathbf{X}}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\
&= \mathcal{M}_{\mathbf{X}}\mathbf{u}. && (\mathcal{M}_{\mathbf{X}}\mathbf{X} = \mathbf{0}_n)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{u}}^T\hat{\mathbf{u}} &= \left(\mathcal{M}_{\mathbf{X}}\mathbf{u}\right)^T\left(\mathcal{M}_{\mathbf{X}}\mathbf{u}\right) \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}^T\mathcal{M}_{\mathbf{X}}\mathbf{u} \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}^2\mathbf{u} && \text{(Symmetric)} \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}\mathbf{u}. && \text{(Idempotent)}
\end{aligned}$$

8. This is a problem using R.

(a) Set up a function in R to determine the singularity of the matrix in each matrix equation below. The function should automatically derive a solution if it is non-singular, or should print "It does not have a unique solution'' if a matrix is singular. (use det() function to derive the determinants of matrices)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Answer:

```r
# (a)

# Create a function that solves a system of linear equations
solve_sys_lin <- function(A, b) {
  ifelse(abs(det(A)) >= 10^(-6),                    # set up tolerance value
         return(matlib::inv(A) %*% b),              # if not singular
         return(print("It does not have a unique solution."))  # if singular
         )
}

## 1st system of linear equations
vt1 <- c(1, -2, 1, 0, 2, -8, -4, 5, 9)
vt2 <- c(0, 8, -9)
mtrx1 <- matrix(vt1, nrow = 3, ncol = 3, byrow = T)
mtrx2 <- matrix(vt2, nrow = 3, ncol = 1, byrow = T)
solve_sys_lin(mtrx1, mtrx2)
```

```
##      [,1]
## [1,]   29
## [2,]   16
## [3,]    3
```

```r
## 2nd system of linear equations
vt3 <- c(1, 4, 2, 2, 5, 1, 3, 6, 0)
vt4 <- c(1, -2, 3)
mtrx3 <- matrix(vt3, nrow = 3, ncol = 3, byrow = T)
mtrx4 <- matrix(vt4, nrow = 3, ncol = 1, byrow = T)
solve_sys_lin(mtrx3, mtrx4)
```

```
## [1] "It does not have a unique solution."
```

(b) Create another function including a for loop to derive the determinant of the matrix using cofactor(), which means this time you should use Laplace Expansion instead of det(). Compare it with the value derived from det() function from each matrix equation in (a).

```
# (b)
det_cofactor <- function(A) {
  det1 <- 0
  for (c in 1:ncol(A))
      det1 = det1 + A[1, c] * cofactor(A, 1, c)
  return(det1)
}

## 1st system of linear equations
det_cofactor(mtrx1)
```

```
## [1] 2
```

```
print(det(mtrx1))
```

```
## [1] 2
```

```
## 2nd system of linear equations
det_cofactor(mtrx3)
```

```
## [1] 0
```

```
print(det(mtrx3))
```

```
## [1] -1.332268e-15
```