Math Camp 2024, UCSB
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This problem set will help you review the key concepts from the course so far. You are asked to submit this assignment using LaTeX or R markdown (only for the Linear Algebra problem sets). The purpose is to give you an opportunity to use LaTeX and R Markdown if you haven't. Please have only one member in your group send an email to the instructor by the due date with other members cc'ed in the email.

1. Consider the following matrices:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) Check $(\mathcal{AB})^T = \mathcal{B}^T\mathcal{A}^T$.
   (b) Check $(\mathcal{AB})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$.
   (c) Check $(\mathcal{A}^T)^{-1} = (\mathcal{A}^{-1})^T$.

2. Let A and B be $n \times n$ matrices, where n is a positive integer $(n \geq 1)$. Prove whether the following is true or false:

$$\det(A+B) = \det(A) + \det(B)$$

3. For each function below, determine whether it is convex or concave using its Hessian matrix.

   (a) $f(x,y) = -x^2 - y^2 + xy + 2x - y$
   (b) $g(a,b,c) = 3a^2 + 2b^2 + c^2 - 2ab + 2bc - 6a - 4b - 2c$

4. Let X be a $n \times k$ matrix with $\text{rank}(\mathbf{X}) = k (n \geq k)$. An annihilator matrix $\mathcal{M}_{\mathbf{X}}$ is

$$\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

   Show that $\mathcal{M}_{\mathbf{X}}$ is symmetric and idempotent. Clarify which property you are using in each step.

5. Find the eigenvalues and eigenvectors of the following matrix. Normalize the norm to one.

$$\mathcal{C} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$$

6. Express the following terms

   a. $\sum_{i=1}^{n} \frac{u_i^2}{2\sigma^2}$

   b. $\sum_{i=1}^{n} \lambda_i u_i^2$

   into quadratic forms using the following notations:

$$\mathbf{u} := [u_1 \ \cdots \ u_n]', \quad \Sigma := \sigma^2 \mathcal{I}_n, \quad \text{and} \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

7. Consider the following regression equation where $n > k$:

$$\begin{aligned} y_i &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ik-1}\beta_{k-1} + \beta_k + u_i & \forall\ i = 1, \cdots, n \\ &= \mathbf{x}_i{}^T \boldsymbol{\beta} + u_i & \forall\ i = 1, \cdots, n \end{aligned}$$

   We can rewrite this as a matrix equation

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

   where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

   (a) Check the following: $\mathbf{X}^T\mathbf{X} = \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T$ and $\mathbf{X}^T\mathbf{y} = \sum_{i=1}^{n} \mathbf{x}_i y_i$.

   (b) Assume that $\text{rank}(\mathbf{X}) = k$. We derived $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ in class. Show that

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i\right).$$

   (c) Show that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T\right)^{-1} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \mathbf{u}_i\right)$$

   (d) Let $\hat{\mathbf{u}} := \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$. Show that $\sum_{i=1}^{n} \hat{u}_i^2 = \mathbf{u}^T \mathcal{M}_{\mathbf{X}} \mathbf{u}$ where $\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$.
   Note: You may start from $\hat{\mathbf{u}} = \mathcal{M}_{\mathbf{X}}\mathbf{y} = \mathcal{M}_{\mathbf{X}}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u})$.

8. This is a problem using R.
Set up a function in R to determine the singularity of the matrix in each matrix equation below. The function should automatically derive a solution if it is non-singular, or should print "It does not have a unique solution" if a matrix is singular. (use `det()` function to derive the determinants of matrices)

(a)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$